Available at WWW.MATHEMATICSWEB.ORG POWERED BY SCIENCE DIRECT
Journal of Approximation Theory 121 (2003) 244–268
JOURNAL OF Approximation Theory
http://www.elsevier.com/locate/jat

Entropy numbers of Sobolev embeddings of radial Besov spaces

Thomas Kühn,[a] Hans-Gerd Leopold,[b,1,*] Winfried Sickel,[b,2] and Leszek Skrzypczak[c]

[a] *Mathematisches Institut, Universität Leipzig, Augustusplatz 10/11, D-04109 Leipzig, Germany*
[b] *Mathematisches Institut, Friedrich-Schiller-Universität, Ernst-Abbe-Platz 1-4, D-07743 Jena, Germany*
[c] *Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Matejki 48-49, 60-769 Poznań, Poland*

Received 6 December 2001; accepted 22 November 2002
Communicated by Peter Oswald

Abstract

Let $RB^s_{p,q}(\mathbb{R}^n)$ be the radial subspace of the Besov space $B^s_{p,q}(\mathbb{R}^n)$. We prove the independence of the asymptotic behavior of the entropy numbers

$$e_k(\mathrm{id} : RB^{s_0}_{p_0,q_0}(\mathbb{R}^n) \mapsto RB^{s_1}_{p_1,q_1}(\mathbb{R}^n))$$

from the difference $s_0 - s_1$ as long as the embedding itself $RB^{s_0}_{p_0,q_0}(\mathbb{R}^n) \hookrightarrow RB^{s_1}_{p_1,q_1}(\mathbb{R}^n)$ is compact. In fact, we shall show that

$$e_k(\mathrm{id} : RB^{s_0}_{p_0,q_0}(\mathbb{R}^n) \mapsto RB^{s_1}_{p_1,q_1}(\mathbb{R}^n)) \sim k^{-n(\frac{1}{p_0}-\frac{1}{p_1})}.$$

This is in a certain contrast to earlier results on entropy numbers in the context of Besov spaces $B^s_{p,q}(\Omega)$ on bounded domains Ω.

Keywords: Radial spaces; Weighted spaces; Atomic decompositions; Entropy numbers; Traces; Compact embeddings

*Corresponding author.
E-mail addresses: kuehn@mathematik.uni-leipzig.de (T. Kühn), leopold@minet.uni-jena.de (H.-G. Leopold), sickel@minet.uni-jena.de (W. Sickel), lskrzyp@amu.edu.pl (L. Skrzypczak).
[1] Partly supported by DFG-contract Tr 374/4.
[2] Partly supported by DFG-contract SCHM 969/3-1.

0021-9045/03/$ - see front matter © 2003 Elsevier Science (USA). All rights reserved.
doi:10.1016/S0021-9045(02)00063-1

1. Introduction

The aim of the paper is to investigate the quality of compact embeddings between the radial subspaces of Besov and Triebel–Lizorkin spaces on the Euclidean n-space $\mathbb{R}^n$. We do not have in mind specific applications but rather want to direct the attention to a new phenomenon. Let $B^s_{p,q}(\Omega)$ be the Besov space defined on the bounded domain Ω in $\mathbb{R}^n$. Then the embedding

$$B^{s_0}_{p_0,q_0}(\Omega) \hookrightarrow B^{s_1}_{p_1,q_1}(\Omega) \quad (s_1 < s_0, p_0 < p_1)$$

is compact if and only if $\delta = s_0 - s_1 - (n/p_0 - n/p_1) > 0$. One can measure the quality of embeddings in terms of the asymptotic behavior of the corresponding entropy numbers. In this situation it is known that

$$e_k(\mathrm{id} : B^{s_0}_{p_0,q_0}(\Omega) \mapsto B^{s_1}_{p_1,q_1}(\Omega)) \sim k^{-\frac{s_0 - s_1}{n}},$$

cf. [7] (here $a \sim b$ indicates a two-sided estimate). The parameters p_0 and p_1 disappear on the right-hand side, only the difference $s_0 - s_1$ matters. In case of the radial subspaces $RB^s_{p,q}(\mathbb{R}^n)$ of the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ we meet the converse situation. Again the embedding

$$RB^{s_0}_{p_0,q_0}(\mathbb{R}^n) \hookrightarrow RB^{s_1}_{p_1,q_1}(\mathbb{R}^n)$$

is compact if and only if $\delta > 0$. But now the entropy numbers behave completely different. We shall prove

$$e_k(\mathrm{id} : RB^{s_0}_{p_0,q_0}(\mathbb{R}^n) \mapsto RB^{s_1}_{p_1,q_1}(\mathbb{R}^n)) \sim k^{-n(\frac{1}{p_0} - \frac{1}{p_1})}.$$

Here the influence of s_0 and s_1 disappears and only the difference in $1/p_0$ and $1/p_1$ counts.

Symmetry as well as weights can be used to generate compactness of embeddings on $\mathbb{R}^n$. That has been known since the seventies, cf. e.g. [6,18] in case of first-order Sobolev spaces. In the general framework of Besov and Triebel–Lizorkin spaces a detailed account has been made in our previous paper [26].

The asymptotic behavior of entropy numbers of Sobolev embeddings is known in some situations, in particular for

- spaces on bounded domains;
- weighted spaces on $\mathbb{R}^n$ (partial results).

As we shall see later entropy numbers of Sobolev embeddings of weighted spaces on $\mathbb{R}^n$ are closely related to our problem here. Roughly speaking, we split the radial subspace of the Besov space into two parts. One part, denoted by $RB^s_{p,q}(\mathbb{R}^n, [1, \infty))$, contains radial distributions which have support in the set $\{x\colon\ |x| \geqslant 1\}$ and the other part, denoted by $RB^s_{p,q}(\mathbb{R}^n, [0, 2])$ consists of radial distributions having support in

$\{x\colon |x| \leqslant 2\}$. For a radial distribution f all information is contained in its trace $\mathrm{tr}^* f$ on $\mathbb{R}_+$. This phrase can be made rigorous if the distribution is sufficiently regular. Let $w_\alpha(x) = (1+|x|^2)^{\alpha/2}, \alpha \in \mathbb{R}$. It turns out that the mapping tr^* becomes an isomorphism of

$$RB^s_{p,q}(\mathbb{R}^n, [1,\infty)) \quad \text{onto} \quad B^s_{p,q}(\mathbb{R}_+, w_{\frac{(n-1)}{p}}, [1,\infty)),$$

where the latter space is a weighted Besov space defined on $\mathbb{R}_+$, cf. Section 2.3 for further details.

Now, consider the pair $RB^{s_0}_{p_0,q_0}(\mathbb{R}^n)$ and $RB^{s_1}_{p_1,q_1}(\mathbb{R}^n)$, where $p_0 < p_1$ and $s_0 - s_1 \geqslant n(1/p_0 - 1/p_1)$. In order to establish the estimates from below we employ the above remarks and reduce them to the study of the estimates from below with respect to the pair

$$B^{s_0}_{p_0,q_0}(\mathbb{R}_+, w_{\frac{(n-1)}{p_0}}, [1,\infty)) \quad \text{and} \quad B^{s_1}_{p_1,q_1}(\mathbb{R}_+, w_{\frac{(n-1)}{p_1}}, [1,\infty)).$$

To prove the upper estimates, we will use a further tool-the radial φ-transform of Epperson and Frazier. This transform will allow us to reduce the upper estimate for the entropy numbers to upper estimates of entropy numbers of certain sequence spaces in a very convenient way. Estimates of entropy numbers of the associated sequence spaces will be the main subject in Section 3. In the final section we prove our main result announced above and formulate also a parallel result for the Lizorkin–Triebel scale.

The described situation with $p_0 < p_1$ and $s_0 - s_1 \geqslant n(1/p_0 - 1/p_1)$ is very similar to that in the weighted case on the interval $[1,\infty)$. Assuming the dominance of the behavior near infinity over the behavior around the origin, our result could have been deduced from the two-sided estimate

$$e_k(\mathrm{id} : B^{s_0}_{p_0,q_0}(\mathbb{R}_+, w_{\frac{(n-1)}{p_0}}, [1,\infty)) \mapsto B^{s_1}_{p_1,q_1}(\mathbb{R}_+, w_{\frac{(n-1)}{p_1}}, [1,\infty)) \sim k^{-n(\frac{1}{p_0}-\frac{1}{p_1})}.$$

However, restricted to the situation of interest here, the upper estimate in this asymptotic relation is still open [13, Conjecture 2.5], cf. also [14, Theorem 4.2] or [7, Theorem 4.3.2], and we could not use it for an alternative approach.

2. Relations between sequence and function spaces

Our main aim consists in a characterization of the asymptotic behavior of the entropy numbers of the Sobolev embeddings. For the corresponding estimates the following tools are essential:

(a) the entropy numbers of embeddings of certain weighted sequence spaces;
(b) some identifications between function spaces and sequence spaces.

The described method is standard in that field and employed in many places, cf. e.g. the monographs [7,29] or most recently the papers [5,17]. Here we will work with the following weighted sequence spaces:

$$\ell_q(2^{j\delta}\ell_{p,w_\alpha}) = \Big\{(s_{j,k})_{j,k}\colon \|s_{j,k}|\ell_q(2^{j\delta}\ell_{p,w_\alpha})\| = \left(\sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty}2^{j\delta p}(1+k)^{\alpha}|s_{j,k}|^p\right)^{q/p}\right)^{1/q} < \infty\Big\}, \tag{1}$$

where $\alpha > 0$ is a real number, $\delta \geqslant 0$ and $1 \leqslant p, q \leqslant \infty$ (with the usual modification if $p = \infty$ or $q = \infty$). If $\delta = 0$ we will write $\ell_q(\ell_{p,w_\alpha})$. We postpone the estimates of the entropy numbers for the weighted sequence spaces to Section 3 and start with a description of some mappings which connect the function spaces and the sequence spaces.

2.1. The radial φ-transform

We recall the Epperson–Frazier construction of the radial φ-transform, cf. [8]. We will do this in some detail because we are going to use it with a different normalization.

Let φ, $\psi \in \mathcal{S}(\mathbb{R}^n)$ be radial functions such that $\operatorname{supp}\hat{\varphi}, \hat{\psi} \subset \{\xi\colon \frac{1}{4} < |\xi| < 1\}$, $|\hat{\varphi}(\xi)|, |\hat{\psi}(\xi)| \geqslant c > 0$ if $\frac{3}{10} \leqslant |\xi| \leqslant \frac{5}{6}$. Let $\Phi, \Psi \in \mathcal{S}(\mathbb{R}^n)$ be radial functions satisfying $\operatorname{supp}\hat{\Phi}, \hat{\Psi} \subset \{\xi\colon |\xi| < 1\}$, $|\hat{\Phi}(\xi)|, |\hat{\Psi}(\xi)| \geqslant c > 0$ if $|\xi| \leqslant \frac{5}{6}$. We may assume that the above functions satisfy the following identity:

$$\overline{\hat{\Phi}}(\xi)\hat{\Psi}(\xi) + \sum_{j=1}^{\infty}\overline{\hat{\varphi}}(2^{-j}\xi)\hat{\psi}(2^{-j}\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^n,$$

cf. [8]. We put $\varphi_j(x) = 2^{jn}\varphi(2^j x)$ and $\psi_j(x) = 2^{jn}\psi(2^j x)$, $j = 1, 2, \ldots$. Let $d\sigma_t$ denote the usual surface measure on the sphere of radius t (not normalized) and let ω_{n-1} denote the measure of the unit sphere. Then we have $\int d\sigma_t = \omega_{n-1}t^{n-1}$. Let J_v denote the Bessel function of order v. As the definition we take

$$J_v(x) = \begin{cases} \dfrac{(x/2)^v}{\sqrt{\pi}\Gamma(v+\frac{1}{2})}\int_{-1}^{1}(1-t^2)^{v-1/2}e^{ixt}\,dt & \text{if } v > -\dfrac{1}{2},\\[2ex] \left(\dfrac{2}{\pi x}\right)^{1/2}\cos x & \text{if } v = -\dfrac{1}{2}. \end{cases}$$

Let $\mu_{v,1} < \mu_{v,2} < \cdots$ be the positive zeros of J_v. For $k = 1, 2, \ldots$ we put

$$\varphi_{j,k}^{(s,p)}(x) = 2^{j(s+\frac{n}{2}-\frac{n}{p})}k^{\frac{1-n}{2}} \times \left(\frac{2^{j(n-2)+1}}{\mu_{v,k}^n J_{v+1}^2(\mu_{v,k})\omega_{n-1}}\right)^{1/2}(\varphi_j * d\sigma_{2^{-j}\mu_{v,k}})(x), \quad j = 1, 2, 3, \ldots,$$

$$\psi_{j,k}^{(s,p)}(x) = 2^{-j(s+\frac{n}{2}-\frac{n}{p})} k^{\frac{n-1}{2}} \times \left(\frac{2^{j(n-2)+1}}{\mu_{v,k}^n J_{v+1}^2(\mu_{v,k})\omega_{n-1}} \right)^{1/2} (\psi_j * d\sigma_{2^{-j}\mu_{v,k}})(x), \quad j = 1, 2, 3, \dots,$$

$$\varphi_{0,k}^{(s,p)}(x) = k^{\frac{1-n}{2}} \left(\frac{2}{\mu_{v,k}^n J_{v+1}^2(\mu_{v,k})\omega_{n-1}} \right)^{1/2} (\Phi * d\sigma_{\mu_{v,k}})(x),$$

$$\psi_{0,k}^{(s,p)}(x) = k^{\frac{n-1}{2}} \left(\frac{2}{\mu_{v,k}^n J_{v+1}^2(\mu_{v,k})\omega_{n-1}} \right)^{1/2} (\Psi * d\sigma_{\mu_{v,k}})(x).$$

Observe, if the differential dimensions $s_0 - \frac{n}{p_0}$ and $s_1 - \frac{n}{p_1}$ coincide then $\varphi_{j,k}^{(s_0,p_0)} = \varphi_{j,k}^{(s_1,p_1)}$ and similar for ψ.

Agreement: From now on we fix v by $v = \frac{n-2}{2}$ and drop the influence of v in notations (as we already did in case of $\varphi_{j,k}^{(s,p)}$ and $\psi_{j,k}^{(s,p)}$).

The functions $\varphi_{j,k}^{(s,p)}$ and $\psi_{j,k}^{(s,p)}$ are radial. Moreover, any radial $f \in \mathcal{S}'(\mathbb{R}^n)$ can be decomposed into

$$f = \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \langle f, \varphi_{j,k}^{(s,p)} \rangle \psi_{j,k}^{(s,p)}$$

(convergence in $\mathcal{S}'(\mathbb{R}^n)$), cf. [8]. We will not give a definition of the Besov spaces here. The reader who is not familiar with the standard properties is referred to [11,20,28,29].

Theorem 1 (Epperson–Frazier). *Let* $s \in \mathbb{R}$ *and* $1 \leqslant p, q \leqslant \infty$. *The operators*

$$S_{(s,p)} : RB_{p,q}^s(\mathbb{R}^n) \to \ell_q(\ell_{p,n-1})$$

and

$$T_{(s,p)} : \ell_q(\ell_{p,n-1}) \to RB_{p,q}^s(\mathbb{R}^n)$$

defined by

$$S_{(s,p)}(f) = (\langle f, \varphi_{j,k+1}^{(s,p)} \rangle)_{j,k}, \tag{2}$$

$$T_{(s,p)}((\gamma_{j,k})) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \gamma_{j,k} \psi_{j,k+1}^{(s,p)} \tag{3}$$

are bounded. Moreover, the operator T *is a retraction, i.e.* $T \circ S = \mathrm{id}$.

Proof. In [8] Epperson and Frazier worked with a different normalization. Put

$$\tilde{\varphi}_{j,k} = \lambda_{j,k}^0 \varphi_{j,k}^{(s,p)}, \quad \lambda_{j,k}^0 = 2^{-j(s+\frac{n}{2}-\frac{n}{p})} k^{\frac{n-1}{2}}$$

and

$$\tilde{\psi}_{j,k} = \lambda^1_{j,k} \psi^{(s,p)}_{j,k}, \quad \lambda^1_{j,k} = 2^{j(s+\frac{n}{2}-\frac{n}{p})} k^{\frac{1-n}{2}}.$$

Let $A_{j,k}$ denote the annulus (a ball if $k = 0$),

$$\{x \in \mathbb{R}^n\colon\ 2^{-j}\mu_k \leqslant |x| \leqslant 2^{-j}\mu_{k+1}\}, \quad \mu_0 = 0.$$

The characteristic functions of these sets are denoted by $\mathscr{X}_{A_{j,k}}$. Following [8] we introduce the following sequence spaces $h^s_{p,q}$ by means of the norm

$$||(s_{j,k})|h^s_{p,q}|| = \left(\sum_{j=0}^{\infty} \left|\left| \sum_{k=0}^{\infty} 2^{js} |s_{j,k}| |A_{j,k+1}|^{-1/2} \mathscr{X}_{A_{j,k+1}} \mid L_p(\mathbb{R}^n) \right|\right|^q \right)^{1/q} < \infty.$$

Then the operators

$$\tilde{S}\colon RB^s_{p,q}(\mathbb{R}^n) \to h^s_{p,q} \quad \text{and} \quad \tilde{T}\colon h^s_{p,q} \to RB^s_{p,q}(\mathbb{R}^n)$$

defined by

$$\tilde{S}(f) = (\langle f, \tilde{\varphi}_{j,k} \rangle)_{j,k+1},$$

$$\tilde{T}((s_{j,k})) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} s_{j,k} \tilde{\psi}_{j,k+1}$$

are bounded. Moreover, the operator $\tilde{T}$ is a retraction, i.e. $\tilde{T} \circ \tilde{S} = \mathrm{id}$. Using McMahon's asymptotic expansion $\mu_k = \mu_{v,k} = (k + \frac{v}{2} - \frac{1}{4})\pi + O(\frac{1}{k})$, cf. [30, pp. 505–506] one derives that

$$|A_{j,k}| \sim 2^{-jn} k^{n-1}.$$

In consequence

$$\begin{aligned}
&\left|\left| \sum_{k=0}^{\infty} 2^{js} |s_{j,k}| |A_{j,k+1}|^{-\frac{1}{2}} \mathscr{X}_{A_{j,k+1}} | L_p(\mathbb{R}^n) \right|\right| \\
&\quad \sim \left(\sum_{k=0}^{\infty} 2^{jps} |s_{j,k}|^p |A_{j,k+1}|^{1-\frac{p}{2}} \right)^{1/p} \\
&\quad \sim \left(\sum_{k=0}^{\infty} |2^{j(s+n(\frac{1}{2}-\frac{1}{p}))} |s_{j,k}| (1+k)^{(1-n)/2}|^p (1+k)^{n-1} \right)^{1/p} \\
&\quad \sim \left(\sum_{k=0}^{\infty} |\gamma_{j,k}|^p (1+k)^{n-1} \right)^{1/p}.
\end{aligned}$$

The assertion of Theorem 1 follows from these equivalences. □

Corollary 1. *Let* $p_0 < p_1$ *and* $s_0 - \frac{n}{p_0} \geqslant s_1 - \frac{n}{p_1}$. *Let* $\delta = s_0 - s_1 + n(\frac{1}{p_1} - \frac{1}{p_0})$ *and* $\tilde{s} = s_1 + n(\frac{1}{p_0} - \frac{1}{p_1})$. *Then*

$$S_{(\tilde{s},p_0)} = S_{(s_1,p_1)} = S, \quad T_{(\tilde{s},p_0)} = T_{(s_1,p_1)} = T \tag{4}$$

and the following diagram is commutative:

$$\begin{array}{ccc} RB^{s_0}_{p_0,q}(\mathbb{R}^n) & \overset{S}{\longrightarrow} & \ell_q(2^{j\delta}\ell_{p_0,n-1}) \\ \mathrm{id}\Big\downarrow & & \Big\downarrow\mathrm{id} \\ RB^{s_1}_{p_1,q}(\mathbb{R}^n) & \overset{T}{\longleftarrow} & \ell_q(\ell_{p_1,n-1}) \end{array}$$

Proof. The identities in (4) follow from the definition of the functions $\varphi^{(s,p)}_{j,k}$ and $\psi^{(s,p)}_{j,k}$ and the definition of the operators. So, for $s_0 = \tilde{s}$ the commutivity of the diagram follows from Theorem 1 and $B^{s_0}_{p_0,q}(\mathbb{R}^n) \hookrightarrow B^{s_1}_{p_1,q}(\mathbb{R}^n)$. Let $s_0 > \tilde{s}$. For $j > 0$ we have

$$\varphi^{(s_0,p_0)}_{j,k} = 2^{j\delta}\varphi^{(\tilde{s},p_0)}_{j,k} \quad \text{and} \quad \psi^{(s_0,p_0)}_{j,k} = 2^{j\delta}\psi^{(\tilde{s},p_0)}_{j,k}.$$

Let $f \in RB^{s_0}_{p_0,q}(\mathbb{R}^n)$. Then

$$S(f) = (\langle f, \varphi^{(\tilde{s},p_0)}_{j,k+1}\rangle)_{j,k} = (2^{-j\delta}\langle f, \varphi^{(s_0,p_0)}_{j,k+1}\rangle).$$

By Theorem 1, the operator S is a bounded operator from the space $RB^{s_0}_{p_0,q}(\mathbb{R}^n)$ into $\ell_q(\ell_{p_0,n-1})$. Moreover, the operator

$$T : \ell_q(2^{j\delta}\ell_{p_1,n-1}) \to RB^{s_1}_{p_1,q}(\mathbb{R}^n)$$

defined by

$$T((\gamma_{j,k})) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \gamma_{j,k}\psi^{(\tilde{s},p_0)}_{j,k+1} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \gamma_{j,k}\psi^{(s_1,p_1)}_{j,k+1}$$

is a bounded retraction. So we get the above commutative diagram. □

Remark 1. Corollary 1 will be used to derive estimates of the entropy numbers from above. Observe that the functions $\psi^{(s,p)}_{j,k}$ are not linearly independent. So different sequences may lead to the same distribution in (2). That makes clear that Theorem 1 cannot be used for the estimates from below.

2.2. Atomic decomposition of weighted spaces

This subsection has a preparatory character. Consider the L_p-case, then it becomes obvious that spaces of radial functions are related to weighted spaces defined on the positive half-line. To prepare a similar statement for Besov spaces we investigate first atomic decompositions for weighted Besov spaces.

We recall the definition of an atom, cf. [11,27] or [29]. For an open set Q and $r>0$ we put $rQ=\{x\in\mathbb{R}^n\colon \operatorname{dist}(x,Q)<r\}$. Observe that Q is always a subset of rQ whatever r is.

Definition 1. Let $s\in\mathbb{R}$ and let $1\leqslant p\leqslant\infty$. Let L and M be integers such that $L\geqslant 0$ and $M\geqslant -1$. Let $Q\subset\mathbb{R}^n$ be an open connected set with $\operatorname{diam} Q=r$.

(a) A smooth function $a(x)$ is called an 1_L-atom centered in Q if

$$\operatorname{supp} a\subset\frac{r}{2}Q,$$
$$\sup_{y\in\mathbb{R}^n}|D^\alpha a(y)|\leqslant 1,\quad |\alpha|\leqslant L.$$

(b) A smooth function $a(x)$ is called an $(s,p)_{L,M}$-atom centered in Q if

$$\operatorname{supp} a\subset\frac{r}{2}Q,$$
$$\sup_{y\in\mathbb{R}^n}|D^\alpha a(y)|\leqslant r^{s-|\alpha|-\frac{n}{p}},\quad |\alpha|\leqslant L,$$
$$\left|\int_{\mathbb{R}^n}a(y)\varphi(y)\,dy\right|\leqslant r^{s+M+1+\frac{n}{p'}}||\varphi|C^{M+1}(\overline{rQ})||,\quad \varphi\in C^\infty(\mathbb{R}^n),$$

where $1/p+1/p'=1$.

Remark 2. If $M=-1$, then the moment condition in part (b) becomes superfluous.

Among others in [26] the authors constructed a regular sequence of coverings with certain special properties which we now recall. Consider the shells (balls if $k=0$)

$$P_{j,k}=\{x\in\mathbb{R}^n\colon k2^{-j}\leqslant|x|<(k+1)2^{-j}\},\quad j=0,1,\ldots,\quad k=0,1,\ldots\ .$$

Then there is a sequence $\{\Omega_j^R\}_{j=0}^\infty=\{\{\Omega_{j,k,\ell}^R\}_{k,\ell}\}_{j=0}^\infty$ of coverings of $\mathbb{R}^n$ such that

(a) all $\Omega_{j,k,\ell}^R$ are balls with centres $y_{j,k,\ell}$ satisfying

$$|y_{j,k,\ell}|=\begin{cases}2^{-j}(k+1/2) & \text{if } k>0,\\ 0 & \text{if } k=0;\end{cases}$$

(b) $P_{j,k}\subset\bigcup_{\ell=1}^{C(n,k)}\Omega_{j,k,\ell}^R$, $j=0,1,\ldots;$

(c) $\operatorname{diam}\Omega_{j,k,\ell}^R=12\cdot 2^{-j}$;

(d) the sums $\sum_{k=0}^\infty\sum_{\ell=1}^{C(n,k)}\mathscr{X}_{j,k,\ell}(x)$ are uniformly bounded in $x\in\mathbb{R}^n$ and $j=0,1,\ldots$ (here $\mathscr{X}_{j,k,\ell}$ denotes the characteristic function of $\Omega_{j,k,\ell}^R$);

(e) $\Omega_{j,k,\ell}^R=\{x\in\mathbb{R}^n\colon 2^jx\in\Omega_{0,k,\ell}^R\}$, $j=0,1,\ldots;$

(f) $C(n,k) \leqslant (2k+1)^{n-1}$, $C(n,0) = 1$.
(g) With an appropriate enumeration it holds

$$\{(x_1, 0, \ldots, 0)\colon\ x_1 \geqslant 0\} \subset \bigcup_{k=0}^{\infty} \Omega^R_{j,k,1}$$

and

$$|\{(x_1, 0, \ldots, 0)\colon\ x_1 \in \mathbb{R}\} \cap \Omega^R_{j,k,1}| \sim 2^{-j}.$$

Here $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}$. For $k = 0$ it makes sense to write $\Omega^R_{j,0,1} = \Omega^R_{j,0}$. This abbreviation has been used in [26] but we do not use it here. Property (g) is not stated explicitly in [26] but it follows immediately from the construction described there.

We collect some properties of these decompositions in connection with Besov spaces. To do so we introduce a further sequence space: we put

$$\ell_q(\ell^*_p) = \{s = (s_{j,k,\ell})_{j,k,\ell}\colon\ ||s|\ell_q(\ell^*_p)|| < \infty\},$$

where

$$||s|\ell_q(\ell^*_p)|| = \left(\sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty}\ \sum_{\ell=1}^{C(n,k)} |s_{j,k,\ell}|^p\right)^{q/p}\right)^{1/q} \tag{5}$$

(usual modification if p or/and $q = \infty$).

(i) Each $f \in B^s_{p,q}(\mathbb{R}^n)$ can be decomposed into

$$f = \sum_{j=0}^{\infty}\ \sum_{k=0}^{\infty}\ \sum_{\ell=1}^{C(n,k)} s_{j,k,\ell}\, a_{j,k,\ell} \quad (\text{convergence in } \mathscr{S}'(\mathbb{R}^n)), \tag{6}$$

where the $a_{j,k,\ell}$ are $(s,p)_{L,M}$-atoms with respect to $\Omega^R_{j,k,\ell}$ $(j \geqslant 1)$, and the $a_{0,k,\ell}$ are 1_L-atoms with respect to $\Omega^R_{0,k,\ell}$.

(ii) Any formal series $\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=1}^{C(n,k)} s_{j,k,\ell}\, a_{j,k,\ell}$ converges in $\mathscr{S}'(\mathbb{R}^n)$ and its limit belongs to $B^s_{p,q}(\mathbb{R}^n)$ if the sequence $s = (s_{j,k,\ell})_{j,k,\ell}$ belongs to $\ell_q(\ell^*_p)$. Further, there exists a universal constant such that

$$\left\|\sum_{j=0}^{\infty}\ \sum_{k=0}^{\infty}\ \sum_{\ell=1}^{C(n,k)} s_{j,k,\ell}\, a_{j,k,\ell} \Big| B^s_{p,q}(\mathbb{R}^n)\right\| \leqslant c||s|\ell_q(\ell^*_p)|| \tag{7}$$

holds for all sequences $s = (s_{j,k,\ell})_{j,k,\ell}$.

(iii) The infimum of the left-hand side in (7) with respect to all admissible representations (6) yields an equivalent norm on $B^s_{p,q}(\mathbb{R}^n)$.

(iv) In [26] we gave an explicit construction of an atomic decomposition satisfying

$$\left(\sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty}\sum_{\ell=1}^{C(n,k)}|s_{j,k,\ell}|^p\right)^{q/p}\right)^{1/q}\leqslant c||f|B^s_{p,q}(\mathbb{R}^n)|| \tag{8}$$

for some constant c independent of f. Such a decomposition we shall call optimal.

(v) If f belongs to the subspace $RB^s_{p,q}(\mathbb{R}^n)$ consisting of radial distributions in $B^s_{p,q}(\mathbb{R}^n)$, then one may arrive at an optimal decomposition (see the previous item) such that $s_{j,k,\ell}=s_{j,k,1}$, $\ell=1,2,\dots,C(n,k)$, cf. [26].

Based on this decomposition we construct now atomic decompositions of some weighted Besov spaces. The weights we are interested in are

$$w_\alpha(x)=(1+|x|^2)^{\alpha/2},\qquad x\in\mathbb{R}^n,\ \ \alpha\in\mathbb{R}. \tag{9}$$

For the definition of $B^s_{p,q}(\mathbb{R}^n,w_\alpha)$ we refer to Schmeißer and Triebel [23, Chapter 5]. Of great service for us will be the fact that $f\mapsto fw_\alpha$ yields an isomorphism of $B^s_{p,q}(\mathbb{R}^n,w_\alpha)$ onto $B^s_{p,q}(\mathbb{R}^n)$. This has been proved by Franke [10], but see also [23, Theorem 5.1.3]. Consequently, any $f\in B^s_{p,q}(\mathbb{R}^n,w_\alpha)$ admits an atomic decomposition of the form

$$fw_\alpha=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=1}^{C(n,k)}s_{j,k,\ell}a_{j,k,\ell}$$

such that (7) and (8) are satisfied with fw_α instead of f. Dividing by w_α we obtain a decomposition like

$$f=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=1}^{C(n,k)}s_{j,k,\ell}b_{j,k,\ell},$$

where the building blocks $b_{j,k,\ell}$ have to satisfy certain inequalities. It is an exercise that the correct choice is the following.

Definition 2. Let $\alpha\in\mathbb{R}$. Let $\{\Omega_j^R\}_{j=0}^{\infty}$ be as above. Suppose $1\leqslant p\leqslant\infty$ and $s>0$. Denote by L a nonnegative integer.

(a) A smooth function $b(x)$ is called a weighted $(1,\alpha)_L$-atom centered in $\Omega^R_{0,k,\ell}$ if

$$\operatorname{supp} b\subset 6\Omega^R_{0,k,\ell},$$

$$\sup_{y\in\mathbb{R}^n}|D^\gamma b(y)|\leqslant w_{-\alpha}(k),\quad |\gamma|\leqslant L.$$

(b) A smooth function $b(x)$ is called a weighted $(s,p,\alpha)_L$-atom centered in $\Omega^R_{j,k,\ell}$ if

$$\operatorname{supp} b \subset 6 \cdot (2^{-j}) \Omega^R_{j,k,\ell},$$

$$\sup_{y \in \mathbb{R}^n} |D^\gamma b(y)| \leqslant w_{-\alpha}(2^{-j}k)\, (6 \cdot (2^{-j}))^{s-|\gamma|-\frac{n}{p}}, \quad |\gamma| \leqslant L.$$

The atoms defined above depend on the scale and on the place. These new features are, of course, undesirable. However, the above definition is justified by the following proposition.

Proposition 1. *Let $\alpha \in \mathbb{R}$. Let $\{\Omega^R_j\}_{j=0}^\infty$ the particular decomposition described above. Suppose $1 \leqslant p \leqslant \infty$, $s > 0$, and denote by L a natural number satisfying $L > s$.*

(i) *Each $f \in B^s_{p,q}(\mathbb{R}^n, w_\alpha)$ can be decomposed into*

$$f = \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{\ell=1}^{C(n,k)} s_{j,k,\ell} b_{j,k,\ell} \quad (\textit{convergence in } \mathscr{S}'(\mathbb{R}^n)), \tag{10}$$

where the $b_{j,k,\ell}$ are weighted $(s,p,\alpha)_L$-atoms with respect to $\Omega^R_{j,k,\ell}$ $(j \geqslant 1)$ and the $b_{0,k,\ell}$ are weighted $(1,\alpha)_L$-atoms with respect to $\Omega^R_{0,k,\ell}$.

(ii) *If the sequence $s = (s_{j,k,\ell})_{j,k,\ell}$ belongs to $\ell_q(\ell^*_p)$, then the formal series on the right-hand side of* (10) *converges in $\mathscr{S}'(\mathbb{R}^n)$ and*

$$\left\| \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{\ell=1}^{C(n,k)} s_{j,k,\ell} b_{j,k,\ell} | B^s_{p,q}(\mathbb{R}^n, w_\alpha) \right\| \leqslant c \left(\sum_{j=0}^\infty \left(\sum_{k=0}^\infty \sum_{\ell=1}^{C(n,k)} |s_{j,k,\ell}|^p \right)^{q/p} \right)^{1/q} \tag{11}$$

holds with a constant c independent of the sequence $s = (s_{j,k,\ell})_{j,k,\ell}$. Moreover, the infimum with respect to all admissible representations yields an equivalent norm in $B^s_{p,q}(\mathbb{R}^n, w_\alpha)$.

Proof. One only needs to check, that the quotient $a_{j,k,\ell}/w_\alpha$ yields a weighted $(s,p,\alpha)_L$-atom and vice versa, that the product $w_\alpha b_{j,k,\ell}$ yields an $(s,p)_{L,-1}$-atom $(j \geqslant 1)$. □

Remark 3. In [24] Schott gave a different atomic decomposition for $B^s_{p,q}(\mathbb{R}^n, w_\alpha)$. There the atoms are defined as in the unweighted case and the sequence spaces are modified. Of course, one can switch from his decomposition to our one and vice versa (simply by a renormalization of the atoms).

2.3. Traces of spaces of radial functions

To attack the estimate from below we follow a simple philosophy: all information about a radial function is contained in its trace onto the positive half line (at least if the function is sufficiently regular). The rest of this subsection will be used to make this observation rigorous. We define

$$\mathrm{tr}\ : f(x_1, x_2, \ldots, x_n) \to f(x_1, 0, \ldots, 0)$$

and

$$\mathrm{tr}^* : f(x_1, x_2, \ldots, x_n) \to f(t, 0, \ldots, 0), \quad |x_1| = t.$$

Here we restrict ourselves to a study of tr and tr* in the framework of $\mathcal{S}'(\mathbb{R})$. To begin we recall the classical statement about traces of Besov spaces, cf. e.g. [1, Chapter 5]; [20, Appendix]; [28, 2.7].

Suppose $1 \leqslant p, q \leqslant \infty$ and $s > (n-1)/p$. Then tr maps $B^s_{p,q}(\mathbb{R}^n)$ onto $B^{s-\frac{n-1}{p}}_{p,q}(\mathbb{R})$.

If we restrict the consideration to the subspaces $RB^s_{p,q}(\mathbb{R}^n)$, then the situation is not improved in general. There are simple examples to explain this. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be a radial function such that $\psi(x) = 1$ if $|x| \leqslant 1$ and $\psi(x) = 0$ if $|x| \geqslant 2$. Then, if $s > 0, \alpha \neq 2k, k \in \mathbb{N}_0$ we have

$$f_\alpha(x) = \psi(x)|x|^\alpha \in B^s_{p,\infty}(\mathbb{R}^n) \Leftrightarrow s \leqslant \frac{n}{p} + \alpha,$$

cf. e.g. [23, 2.3.1]. If we compare the n-dimensional case with the situation for $n = 1$, then it becomes obvious that the loss in the regularity by taking the trace is as in the general case. Investigating such examples as above makes it clear that the origin is in a certain sense a singular point. For us it will be sufficient to investigate the trace out of the origin. In this case the local regularity of a radial function $f \in RB^s_{p,q}(\mathbb{R}^n)$ is much higher than that one of an arbitrary element of $B^s_{p,q}(\mathbb{R}^n)$. This phenomenon has been observed earlier in [18] in the framework of Sobolev spaces and in [26] for Besov spaces themselves. To make clear what we mean by *out of the origin* we shall use the following notions.

Definition 3. Let $0 < t < \infty$, $1 \leqslant p, q \leqslant \infty$, and $s, \alpha \in \mathbb{R}$.

(i) Then we put

$$B^s_{p,q}(\mathbb{R}^n, w_\alpha, [t, \infty)) = \{f \in B^s_{p,q}(\mathbb{R}^n, w_\alpha): \ \mathrm{supp}\, f \subset \{x \in \mathbb{R}^n: \ |x| \geqslant t\}\}.$$

(ii) We define

$$RB^s_{p,q}(\mathbb{R}^n, [t, \infty)) = \{f \in RB^s_{p,q}(\mathbb{R}^n): \ \mathrm{supp}\, f \subset \{x \in \mathbb{R}^n: \ |x| \geqslant t\}\}$$

and similarly

$$RB^s_{p,q}(\mathbb{R}^n, [0, t]) = \{f \in RB^s_{p,q}(\mathbb{R}^n): \ \mathrm{supp}\, f \subset \{x \in \mathbb{R}^n: \ |x| \leqslant t\}\}.$$

(iii) In the one-dimensional situation we introduce

$$B^s_{p,q}(\mathbb{R}_+, w_\alpha, [t, \infty)) = \{f \in B^s_{p,q}(\mathbb{R}, w_\alpha) : \ \operatorname{supp} f \subset [t, \infty)\}.$$

All types of spaces will be equipped with the natural norm.

Some technicalities. As usual, tr and tr* are well defined for continuous functions. The extension to general functions is then done by a continuity argument along an estimate like

$$\|\operatorname{tr} f|B\| \leqslant c\|f|A\|$$

valid for all sufficiently smooth functions $f \in A$ and for some constant c independent of f. This works well as long as smooth functions are dense. In our case this is true for $q < \infty$. If $q = \infty$ we make use of the so-called Fatou property. Let $f_j, \ j = 1, 2, \ldots$ be a sequence of functions, which has a limit f in $\mathscr{S}'$ and satisfies $\|f_j|B^s_{p,q}\| \leqslant C < \infty$. Then the limit itself belongs to $B^s_{p,q}$ and satisfies

$$\|f|B^s_{p,q}\| \leqslant cC \liminf \|f_j|B^s_{p,q}\|$$

for some constant c independent of f and f_j.

First we investigate the situation out of the origin which is in fact also the more interesting one. For this we shall need the following lemma.

Lemma 1. *Let* $0 < t, \ \lambda < \infty$, $1 \leqslant p, q \leqslant \infty$ *and* $s > 0$. *Let* $\alpha \in \mathbb{R}$.

(i) *The mapping* $f \mapsto f(\lambda\cdot)$ *yields an isomorphism of* $B^s_{p,q}(\mathbb{R}^n, w_\alpha, [t, \infty))$ *onto* $B^s_{p,q}(\mathbb{R}^n, w_\alpha, [t/\lambda, \infty))$.

(ii) *The mapping* $f \mapsto f(\lambda\cdot)$ *yields an isomorphism of* $B^s_{p,q}(\mathbb{R}_+, w_\alpha, [t, \infty))$ *onto* $B^s_{p,q}(\mathbb{R}_+, w_\alpha, [t/\lambda, \infty))$.

Proof. The claim follows from well-known estimates of the dilation operator in the unweighted situation, cf. e.g. [28, 3.4.1], the quoted isomorphism between weighted and unweighted spaces and the fact that $w_\alpha(\lambda x)/w_\alpha(x)$ is a pointwise multiplier for $B^s_{p,q}(\mathbb{R}^n)$, cf. e.g. [22, 4.7]. We sketch a proof. Starting point is the following equivalent quasi-norm in $B^s_{p,q}(\mathbb{R}^n)$ $(M > s > 0)$:

$$\|f|B^s_{p,q}(\mathbb{R}^n)\|^* = \|f|L_p(\mathbb{R}^n)\| + \left(\int [|h|^{-s}\|\Delta^M_h f|L_p(\mathbb{R}^n)\|]^q \frac{dh}{|h|^n} \right)^{1/q}.$$

Hence

$$\begin{aligned}
&\|f(\lambda\cdot)|B^s_{p,q}(\mathbb{R}^n)\|^* \\
&\quad = \lambda^{-n/p}\|f|L_p(\mathbb{R}^n)\| + \lambda^{s-n/p}\left(\int [|h|^{-s}\|\Delta^M_h f|L_p(\mathbb{R}^n)\|]^q \frac{dh}{|h|^n} \right)^{1/q}
\end{aligned}$$

and

$$\lambda^{-n/p}\min(1,\lambda^s)\|f|B^s_{p,q}(\mathbb{R}^n)\|^* \leqslant \|f(\lambda\cdot)|B^s_{p,q}(\mathbb{R}^n)\|^* \leqslant \lambda^{-n/p}\max(1,\lambda^s)\|f|B^s_{p,q}(\mathbb{R}^n)\|^*.$$

Further, in the weighted situation we find

$$\begin{aligned}\|f(\lambda\cdot)w_\alpha|B^s_{p,q}(\mathbb{R}^n)\|^* &= \left\|f(\lambda\cdot)w_\alpha(\lambda\cdot)\frac{w_\alpha(\cdot)}{w_\alpha(\lambda\cdot)}|B^s_{p,q}(\mathbb{R}^n)\right\|^*\\ &\leqslant c\left\|\frac{w_\alpha(\cdot)}{w_\alpha(\lambda\cdot)}|B^s_{\infty,q}(\mathbb{R}^n)\right\|^*\|f(\lambda\cdot)w_\alpha(\lambda\cdot)|B^s_{p,q}(\mathbb{R}^n)\|^*\\ &\leqslant c_{\lambda,\alpha}\|f(\lambda\cdot)w_\alpha(\lambda\cdot)|B^s_{p,q}(\mathbb{R}^n)\|^*,\end{aligned}$$

where $c_{\lambda,\alpha}$ is a finite constant independent of f (for the assertion on pointwise multipliers, cf. e.g. [22, 4.7.1]). The desired result follows from the estimates in the unweighted case and in addition from the same chain of inequalities with w_α replaced by $w_{-\alpha}$. □

Theorem 2. *Suppose* $n\geqslant 2$, $0<t<\infty$, $1\leqslant p,q\leqslant\infty$, *and* $s>0$.

(i) *The operator* tr^* *maps* $RB^s_{p,q}(\mathbb{R}^n,[t,\infty))$ *continuously onto* $B^s_{p,q}(\mathbb{R}_+,w_{\underline{(n-1)}/p},[t,\infty))$.

(ii) *There is a linear and continuous extension operator* ext *which maps* $B^s_{p,q}(\mathbb{R}_+,w_{\underline{(n-1)}/p},[t,\infty))$ *into* $RB^s_{p,q}(\mathbb{R}^n,[t,\infty))$ *and such that* $\mathrm{tr}^*\circ\mathrm{ext}=\mathrm{id}$.

(iii) *The operator* tr^* *is an isomorphism.*

Proof. We shall proof part (i) of the theorem for $t=1$ only. The assertion for general $0<t<\infty$ follows from Lemma 1.

Step 1: *Preparations.* Let $f\in RB^s_{p,q}(\mathbb{R}^n)$ be given by an optimal atomic decomposition

$$f=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=1}^{C(n,k)} s_{j,k}a_{j,k,\ell}, \tag{12}$$

cf. Section 2.2/(iv),(v), and [26] where we have taken $s_{j,k}=s_{j,k,1}$. In addition we assume $\operatorname{supp} f\subset\{x\in\mathbb{R}^n\colon |x|\geqslant 1\}$. Because of $s>0$ the distribution f is regular and the atoms in the above expansion do not need to satisfy a moment condition. Of course, we want to use the additional restriction with respect to the support to derive further information on the coefficients. But we do not have such a localization from the very beginning. So we proceed as follows. Let $\psi\in C_0^\infty(\mathbb{R}^n)$ be a radial cut-off function satisfying $\psi(x)=1$ if $|x|\leqslant 1$ and $\psi(x)=0$ if $|x|\geqslant 2$. Then $\psi(\lambda\cdot)$ is a pointwise multiplier for all spaces $B^s_{p,q}(\mathbb{R}^n)$ and all $\lambda>0$, cf. e.g. [28, 2.8] or [22, 4.7.1].

Hence

$$f(x) = (1-\psi(2x))f(x) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=1}^{C(n,k)} s_{j,k}(1-\psi(2x))a_{j,k,\ell}(x)$$
$$= \sum_{j=0}^{\infty}\sum_{k\geqslant \max(0,2^{j-1}-12)}\sum_{\ell=1}^{C(n,k)} s_{j,k}(1-\psi(2x))a_{j,k,\ell}(x).$$

The functions

$$\tilde{a}_{j,k,\ell}(x) = 2^{-2L}(1-\psi(2x))a_{j,k,\ell}(x)$$

are $(s,p)_{L,-1}$-atoms $(j\geqslant 1)$. Similar $\tilde{a}_{0,k,\ell}$ are 1_L-atoms. In view of Section 2.2/(ii) and (f) this implies

$$\|f|B^s_{p,q}(\mathbb{R}^n)\| \leqslant c\left(\sum_{j=0}^{\infty}\left(\sum_{k\geqslant \max(0,2^{j-1}-12)}(1+k)^{n-1}|s_{j,k}|^p\right)^{q/p}\right)^{1/q}. \tag{13}$$

Step 2: Now, we investigate the trace, assuming that our starting point (12) was an optimal atomic decomposition. To have a well-defined expression for the trace we switch to a partial sum $S^K f$ of the atomic decomposition and ask for the existence of the limit $\mathrm{tr}(S^K f)$ in $\mathscr{S}'(\mathbb{R})$. For $K\in\mathbb{N}$ we put

$$S^K f = \sum_{j=0}^{K}\sum_{k=\max(0,2^{j-1}-12)}^{K}\sum_{\ell=1}^{C(n,k)} s_{j,k}\tilde{a}_{j,k,\ell}.$$

This is a continuous function. For fixed j and k the number of atoms $\tilde{a}_{j,k,\ell}$ which are not identically zero on the x_1-axis is limited by, say M. This number is independent of j and k. Counting these atoms in an appropriate way we arrive at

$$\mathrm{tr}(S^K f)(t) = \sum_{\ell=1}^{M}\left(\sum_{j=0}^{K}\sum_{k=\max(0,2^{j-1}-12)}^{K} s_{j,k}\tilde{a}_{j,k,\ell}(t,0,\ldots,0)\right), \quad t\in\mathbb{R}. \tag{14}$$

The functions

$$b_{j,k,\ell}(t) = (1+k)^{\frac{1-n}{p}}\tilde{a}_{j,k,\ell}(t,0,\ldots,0), \quad k\geqslant \max(0,2^{j-1}-12)$$

are weighted $(s,p,(n-1)/p)_L$-atoms with respect to the intersections of $\Omega^R_{j,k,1}$ with the x_1-axis, cf. property (g) of $(\Omega^R_{j,k,\ell})_{j,k,\ell}$. Here we really need the restriction $k\geqslant \max(0,2^{j-1}-12)$. Similarly the functions

$$b_{0,k,\ell}(t) = (1+k)^{\frac{1-n}{p}}\tilde{a}_{0,k,\ell}(t,0,\ldots,0), \quad k=0,1,\ldots$$

are weighted $(1,(n-1)/p)_L$-atoms with respect to the intersections of $\Omega^R_{0,k,\ell}$ with the x_1-axis. By means of Proposition 1 we obtain

$$||\mathrm{tr}(S^K f)|B^s_{p,q}(\mathbb{R}, w_{\frac{(n-1)}{p}})|| \leqslant c\left(\sum_{j=0}^{K}\left(\sum_{k=\max(0,2^{j-1}-12)}^{K}(1+k)^{n-1}|s_{j,k}|^p\right)^{q/p}\right)^{1/q}, \tag{15}$$

where c does not depend on K and the sequence $\{s_{j,k}\}_{j,k}$. Since we started with an optimal decomposition of f, the right-hand side is dominated by $c||f|B^s_{p,q}(\mathbb{R}^n)||$. This proves the continuity of tr $: RB^s_{p,q}(\mathbb{R}^n,[1,\infty)) \mapsto B^s_{p,q}(\mathbb{R}, w_{(n-1)/p})$. To switch from tr to tr* it is sufficient to note that for a function $g \in B^s_{p,q}(\mathbb{R}, w_\alpha)$ with supp $g \subset [1,\infty)$ and f its even extension we have

$$||f|B^s_{p,q}(\mathbb{R}, w_\alpha)|| \sim ||g|B^s_{p,q}(\mathbb{R}, w_\alpha)||.$$

This proves part (i) in the case $t=1$ up to the claimed surjectivity which will be established in Steps 4 and 5.

Step 3: *The extension operator.* Let us turn back to the adapted atomic decomposition mentioned in the previous subsection. There is an associated sequence of decompositions of unity. An explicit construction has been given in [26], Step 3 of the proof of Theorems 1 and 2. We shall denote these decompositions of unity as there by $\psi_{j,k,\ell}$, which means

$$\sum_{k=0}^{\infty}\sum_{\ell=1}^{C(n,k)}\psi_{j,k,\ell}(x) = 1 \quad \text{for all } x \in \mathbb{R}^n, \quad j = 0,1,\dots,$$

supp $\psi_{j,k,\ell}$ is concentrated near $\Omega^R_{j,k,\ell}$ and $|\mathrm{supp}\,\psi_{j,k,\ell}| \sim |\Omega^R_{j,k,\ell}|$. The second tool we need is the wavelet decomposition with respect to the Daubechies wavelets of sufficiently high order, cf. [2] or [19]. We suppose that the "father" wavelet ϕ belongs to $C^r(\mathbb{R})$ and is compactly supported, cf. [19, Section 3.8]. The associated wavelet ψ shares these properties. The orthonormal basis consists of the functions $\phi(\cdot - k)$, $2^{j/2}\psi(2^j(\cdot)-k)$, $k \in \mathbb{Z}$ and $j=0,1,\dots$. Let C_r be the smallest positive integer such that the supports of the "mother" and "father" wavelets are contained in $[-C_r, C_r]$. We suppose $r > s$. Theorem 4 in [2], more precisely its version for inhomogeneous spaces, implies that any function g from $B^s_{p,q}(\mathbb{R}, w_\alpha)$ can be represented in a unique way by the wavelet expansion

$$g(t)w_\alpha(t) = \sum_{m\in\mathbb{Z}} c_{-1,m}\phi(t-m) + \sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}} c_{j,m}2^{j(1/p-s)}\psi(2^j t - m) \tag{16}$$

and

$$\left(\sum_{m\in\mathbb{Z}}|c_{-1,m}|^p\right)^{1/p} + \left(\sum_{j=0}^{\infty}\left(\sum_{m\in\mathbb{Z}}|c_{j,m}|^p\right)^{q/p}\right)^{1/q} \sim ||g|B^s_{p,q}(\mathbb{R}, w_\alpha)||. \tag{17}$$

Of course,

$$c_{-1,m} = \langle g(y)w_\alpha(y), \phi(y-m)\rangle,$$

$$2^{j(\frac{1}{p}-\frac{1}{2}-s)} c_{j,m} = \langle g(y)w_\alpha(y), 2^{j/2}\psi(2^j y - m)\rangle.$$

We shall assume $g \in B^s_{p,q}(\mathbb{R}, w_{\underline{(n-1)}})$ and $\operatorname{supp} g \subset [C_r, \infty)$. Then our extension operator is defined as follows:

$$\operatorname{ext} g(x) = \sum_{m=1}^{\infty} c_{-1,m} \frac{\phi(|x|-m)}{(1+|x|^2)^{(n-1)/(2p)}} + \sum_{j=0}^{\infty} \sum_{m=(2^j-1)C_r+1}^{\infty} c_{j,m} 2^{j(1/p-s)} \frac{\psi(2^j|x|-m)}{(1+|x|^2)^{(n-1)/(2p)}}. \tag{18}$$

The function $\operatorname{ext} g$ can be rewritten as

$$\begin{aligned}
&\operatorname{ext} g(x)\\
&= \sum_{m=1}^{\infty} c_{-1,m} \frac{\phi(|x|-m)}{(1+|x|^2)^{(n-1)/(2p)}} \left(\sum_{k=0}^{\infty} \sum_{\ell=1}^{C(n,k)} \psi_{0,k,\ell}(x) \right)\\
&\quad + \sum_{j=0}^{\infty} \sum_{m=(2^j-1)C_r+1}^{\infty} c_{j,m} 2^{j(1/p-s)} \frac{\psi(2^j|x|-m)}{(1+|x|^2)^{(n-1)/(2p)}} \left(\sum_{k=0}^{\infty} \sum_{\ell=1}^{C(n,k)} \psi_{j,k,\ell}(x) \right)\\
&= \sum_{m=1}^{\infty} c_{-1,m} \frac{\phi(|x|-m)}{(1+|x|^2)^{(n-1)/(2p)}} \left(\sum_{|k-m| \leqslant C_r+12} \sum_{\ell=1}^{C(n,k)} \psi_{0,k,\ell}(x) \right)\\
&\quad + \sum_{j=0}^{\infty} \sum_{m=(2^j-1)C_r+1}^{\infty} c_{j,m} 2^{j(1/p-s)} \frac{\psi(2^j|x|-m)}{(1+|x|^2)^{(n-1)/(2p)}}\\
&\quad \times \left(\sum_{|k-m| \leqslant C_r+12} \sum_{\ell=1}^{C(n,k)} \psi_{j,k,\ell}(x) \right).
\end{aligned}$$

Observe

$$2^{j(\frac{n}{p}-s)} \frac{\psi(2^j|x|-m)\psi_{j,k,\ell}(x)}{(1+|x|^2)^{(n-1)/(2p)}}, \qquad |k| \leqslant 2^j, \ j \geqslant 1$$

are $(s,p)_{L,-1}$-atoms with respect to $\Omega^R_{j,k,\ell}$ up to an universal constant. Further, the functions

$$2^{-j\frac{n-1}{p}} (1+k)^{\frac{n-1}{p}} 2^{j(\frac{n}{p}-s)} \frac{\psi(2^j|x|-m)\psi_{j,k,\ell}(x)}{(1+|x|^2)^{(n-1)/(2p)}}, \qquad |k| > 2^j, \ j \geqslant 1$$

are $(s,p)_{L,-1}$-atoms with respect to $\Omega^R_{j,k,\ell}$ up to an universal constant. Finally, observe

$$(1+k)^{\frac{n-1}{p}} \frac{\phi(|x|-m)\psi_{0,k,\ell}(x)}{(1+|x|^2)^{(n-1)/(2p)}}$$

are 1_L-atoms with respect to $\Omega^R_{0,k,\ell}$ (again up to a constant). In view of (7) and $C(n,k) \sim (1+k)^{n-1}$ we obtain

$$\|\mathrm{ext}\, g|B^s_{p,q}(\mathbb{R}^n)\| \leqslant c \left(\sum_{j=-1}^{\infty} \left(\sum_{m \geqslant (2^j-1)C_r} |c_{j,m}|^p \right)^{q/p} \right)^{1/q}.$$

This proves the continuity of ext as a mapping from $B^s_{p,q}(\mathbb{R}_+, w_{(n-1)/p}, [C_r, \infty))$ into $RB^s_{p,q}(\mathbb{R}^n)$.

Step 4: Suppose $t > C_r$. The identity $\mathrm{tr}^* \circ \mathrm{ext} = \mathrm{id}$ holds for all C^∞-functions f having compact support. Here we assume $\mathrm{supp}\, f \subset [t, \infty)$. This can be derived from the observation that for functions $f \in B^s_{p,q}(\mathbb{R})$, $s > 1/p$, the expansion (16) converges in the uniform norm. Now a continuity argument with respect to tr* and with respect to ext yields the statement, whenever C_0^∞ is dense, so if $q < \infty$. In case $q = \infty$ we may employ a similar argument. Thanks to $B^s_{p,\infty}(\mathbb{R}) \hookrightarrow B^{s/2}_{p,1}(\mathbb{R})$ we arrive at $\mathrm{tr}^* \circ \mathrm{ext} = \mathrm{id}$ on $B^{s/2}_{p,1}(\mathbb{R})$. But this is enough to guarantee the same conclusion on $B^s_{p,\infty}(\mathbb{R})$.

Step 5: To prove that tr* is an isomorphism we need to show that $\mathrm{tr}^* f = 0$ implies $f = 0$ in the sense of $\mathscr{S}'(\mathbb{R}^n)$. Suppose $f \in RB^s_{p,q}(\mathbb{R}^n, [C_r, \infty))$ and $\mathrm{tr}^* f = 0$. Then there exists a sequence of smooth functions f_j with compact support such that $\mathrm{supp}\, f_j \subset \{x \in \mathbb{R}^n \colon |x| \geqslant C_r\}$ and $\lim_{j\to\infty} f_j = f$ in the sense of $B^{s/2}_{p,1}(\mathbb{R}^n)$. From the boundedness of tr* we obtain $\mathrm{tr}^* f_j \to 0$ if $j \to \infty$ in the norm of $B^{s/2}_{p,1}(\mathbb{R})$. Moreover, $\mathrm{ext} \circ \mathrm{tr}^* f_j = f_j$ by similar reasonings as in Step 4. Now

$$\|(\mathrm{ext} \circ \mathrm{tr}^*) f_j | B^{s/2}_{p,1}(\mathbb{R}^n)\| \leqslant c \|\mathrm{tr}^* f_j | B^s_{p,q}(\mathbb{R})\|$$

which proves $\lim_{j\to\infty} f_j = 0$ (in any case in the sense of $\mathscr{S}'(\mathbb{R}^n)$). □

Remark 4. We make a few remarks to the situation around the origin. Suppose $n \geqslant 2$, $0 < t < \infty$, $1 \leqslant p, q \leqslant \infty$, and $s > (n-1)/p$. Then the operator tr maps $RB^s_{p,q}(\mathbb{R}^n, [0,t])$ continuously into $B^{s-\frac{n-1}{p}}_{p,q}(\mathbb{R}, [0,t])$. That is part of the classical theory of Besov spaces on $\mathbb{R}^n$. But the space $B^{s-\frac{n-1}{p}}_{p,q}(\mathbb{R}, [0,t])$ is too large to be the image of $RB^s_{p,q}(\mathbb{R}^n, [0,t])$ under the mapping tr. That can be made clear by observing that for a function $f \in RB^s_{p,q}(\mathbb{R}^n, [\varepsilon, t])$ we always have $\mathrm{tr}\, f \in B^s_{p,q}(\mathbb{R})$, cf. Theorem 2.

Remark 5. The above approach for investigating the trace extends to spaces with p and q less than 1. In the unweighted situation this has been done in e.g. [11], [15], and

[28, 2.7]. For a complete treatment of all borderline cases see also [9]. In particular, Proposition 1, Lemma 1 and Theorem 2 remain true under the natural restriction $s > n \max(0, \frac{1}{p} - 1)$.

3. Entropy numbers of embeddings of weighted sequence spaces

In the previous section we reduced the function space problem to a sequence space problem, and in this section we estimate the entropy numbers of the relevant sequence space embeddings. The kth (dyadic) entropy number of a bounded linear operator $T : X \to Y$ between two Banach spaces is defined as

$$e_k(T) = \inf\left\{ \varepsilon > 0 \colon \exists y_1, \dots, y_{2^{k-1}} \in Y, T(B_X) \subseteq \bigcup_{j=1}^{2^{k-1}} (y_j + \varepsilon B_Y) \right\},$$

where B_X and B_Y stand for the closed unit balls in X and Y, respectively. For basic properties of entropy numbers and more background we refer to the literature, see e.g. [4,16,21] or [7]. In particular these properties imply that for every pair of Banach spaces X and Y the classes

$$\mathscr{L}^{(e)}_{u,v}(X, Y) = \{T \colon X \to Y | (e_k(T))_k \in \ell_{u,v}\}, \quad 0 < u < \infty, \ 0 < v \leqslant \infty$$

of operators with entropy numbers in the Lorentz sequence space $\ell_{u,v}$ are quasi-Banach spaces with respect to the quasi-norm

$$L^{(e)}_{u,v}(T) = ||(e_k(T))_k | \ell_{u,v}||.$$

In the sequel, we shall exploit the well-known fact that every quasi-norm is equivalent to an r-norm for some r, $0 < r \leqslant 1$.

For our purposes it is enough to consider complex spaces with $1 \leqslant p, q \leqslant \infty, \alpha > 0$ and $\delta \geqslant 0$, like

$$\ell_{p,w_\alpha} = \left\{ (s_k)_k \colon ||s | \ell_{p,w_\alpha}|| = \left(\sum_{k=0}^{\infty} (1+k)^\alpha |s_k|^p \right)^{1/p} < \infty \right\}$$

and

$$\begin{aligned} \ell_q(2^{j\delta}\ell_{p,w_\alpha}) = \Bigg\{ (s_{j,k})_{j,k} \colon & ||s_{j,k} | \ell_q(2^{j\delta}\ell_{p,w_\alpha})|| \\ &= \left(\sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} 2^{j\delta p} (1+k)^\alpha |s_{j,k}|^p \right)^{q/p} \right)^{1/q} < \infty \Bigg\}. \end{aligned}$$

Now we state the main result of this section.

Theorem 3. *Let* $1 \leqslant p_0 < p_1 \leqslant \infty$, $1 \leqslant q_0, q_1 \leqslant \infty$, $\alpha > 0, \delta > 0$, *and set*

$$\beta = (\alpha + 1)(1/p_0 - 1/p_1).$$

Then there are positive constants c and C such that for all $k \in \mathbb{N}$ the estimates

$$ck^{-\beta} \leqslant e_k(\mathrm{id} : \ell_{q_0}(2^{j\delta}\ell_{p_0,w_\alpha}) \to \ell_{q_1}(\ell_{p_1,w_\alpha})) \leqslant Ck^{-\beta}$$

hold.

Proof. *Step* 1: Upper estimate: Let us consider the projections $P_j : \ell_{q_0}(2^{j\delta}\ell_{p_0,w_\alpha}) \to \ell_{p_0,w_\alpha}$ onto the jth vector coordinate, the formal identity $I : \ell_{p_0,w_\alpha} \to \ell_{p_1,w_\alpha}$ and the embedding operators $E_j : \ell_{p_1,w_\alpha} \to \ell_{q_1}(\ell_{p_1,w_\alpha})$, mapping $y \in \ell_{p_1,w_\alpha}$ onto the vector sequence $(0, \dots, y, 0, \dots)$, y being the jth coordinate.

We use the obvious estimates

$$\|P_j\| \leqslant 2^{-j\delta}, \quad \|E_j\| \leqslant 1$$

and the standard decomposition

$$\mathrm{id} = \sum_{j=0}^{\infty} \mathrm{id}_j, \quad \text{where } \mathrm{id}_j = E_j I P_j.$$

For the identity I we have the following commutative diagram:

$$\begin{array}{ccc} \ell_{p_0,w_\alpha} & \overset{D_\mu}{\longrightarrow} & \ell_{p_0} \\ \downarrow I & & \downarrow D_\sigma \\ \ell_{p_1,w_\alpha} & \overset{D_\nu}{\longleftarrow} & \ell_{p_1} \end{array}.$$

Here D_μ, D_σ, D_ν denote the diagonal operators generated by the sequences $\mu_k = (1+k)^{\alpha/p_0}, \sigma_k = (1+k)^{-\alpha(1/p_0-1/p_1)}$ and $\nu_k = (1+k)^{-\alpha/p_1}$, respectively,

$$(D_\sigma x)_l = \sigma_l x_l.$$

Clearly, both D_μ and D_ν are isometries. For D_σ we use the following result due to Carl [3]:

Let $1 \leqslant p_0, p_1 \leqslant \infty,\ 0 < t, u < \infty,\ 0 < v \leqslant \infty$ such that $1/t > 1/p_1 - 1/p_0$ and $1/u = 1/t + 1/p_0 - 1/p_1$. Then

$$\sigma = (\sigma_k)_k \in \ell_{t,v} \quad \text{implies } (e_k(D_\sigma : \ell_{p_0} \to \ell_{p_1}))_k \in \ell_{u,v}.$$

In our special case we have $1/t = \alpha(1/p_0 - 1/p_1)$, $v = \infty$ and $1/u = (\alpha+1)(1/p_0 - 1/p_1) = \beta$, therefore

$$L^{(e)}_{u,\infty}(D_\sigma) = \sup_k k^\beta e_k(D_\sigma) < \infty.$$

Taking into account all previous estimates, the multiplicativity of the entropy numbers yields

$$e_k(\mathrm{id}_j) \leqslant c 2^{-j\delta} k^{-\beta}$$

with a constant c independent of j and k, or, in other words,

$$L^{(e)}_{u,\infty}(\mathrm{id}_j) = \sup_k k^\beta e_k(\mathrm{id}_j) \leqslant c 2^{-j\delta}.$$

Since $L^{(e)}_{u,\infty}$ is equivalent to an r-norm for some r, $0<r\leqslant 1$, we arrive at

$$L^{(e)}_{u,\infty}(\mathrm{id})^r \leqslant c \sum_{j=0}^{\infty} L^{(e)}_{u,\infty}(\mathrm{id}_j)^r \leqslant c \sum_{j=0}^{\infty} 2^{-j\delta r} < \infty,$$

which proves the upper estimate.

Step 2: Lower estimate: For any given $k \in \mathbb{N}$ we consider the following commutative diagram:

$$\begin{array}{ccc} \ell^k_{p_0} & \overset{S}{\longrightarrow} & \ell_{q_0}(2^{j\delta}\ell_{p_0,w_\alpha}) \\ \downarrow \ \mathrm{id} & & \downarrow \ \mathrm{id} \\ \ell^k_{p_1} & \overset{T}{\longleftarrow} & \ell_{q_1}(\ell_{p_1,w_\alpha}) \end{array}.$$

Here the operators S and T are defined by

$$S(\xi_1,\dots,\xi_k) = \begin{cases} \xi_{l+1-k} & \text{if } j=0 \text{ and } k\leqslant l \leqslant 2k-1, \\ 0 & \text{else} \end{cases}$$

and

$$T((s_{j,l})_{j,l}) = (s_{0,k},\dots,s_{0,2k-1}).$$

Using Schütt's result concerning the entropy numbers for embeddings between the finite-dimensional spaces ℓ^M_p—see [25], and again the multiplicativity of the entropy numbers we get, with some constant c independent of k,

$$\begin{aligned} ck^{-(\frac{1}{p_0}-\frac{1}{p_1})} &\leqslant e_k(\mathrm{id} : \ell^k_{p_0} \to \ell^k_{p_1}) \\ &\leqslant \|S\| e_k(\mathrm{id} : \ell_{q_0}(2^{j\delta}\ell_{p_0,w_\alpha}) \to \ell_{q_1}(\ell_{p_1,w_\alpha}))\|T\|. \end{aligned}$$

Together with the obvious relations

$$\|S\| \leqslant (2k)^{\alpha/p_0} \quad \text{and} \quad \|T\| \leqslant (k+1)^{-\alpha/p_1}$$

this implies the lower estimate. □

4. Entropy numbers of id $: RB^{s_0}_{p_0,q_0}(\mathbb{R}^n) \mapsto RB^{s_1}_{p_1,q_1}(\mathbb{R}^n)$

Most of the work is done. A combination of the results obtained in Sections 2 and 3 with a few further known properties of the entropy numbers will prove our main result.

Theorem 4. *Suppose* $1 \leqslant p_0 < p_1 \leqslant \infty$, $1 \leqslant q_0, q_1 \leqslant \infty$, *and* $s_0 - s_1 - n(\frac{1}{p_0} - \frac{1}{p_1}) > 0$. *Let*

$$\beta = n\left(\frac{1}{p_0} - \frac{1}{p_1}\right).$$

Then there exist constants c and C such that

$$ck^{-\beta} \leqslant e_k(\mathrm{id}\ : RB^{s_0}_{p_0,q_0}(\mathbb{R}^n) \mapsto RB^{s_1}_{p_1,q_1}(\mathbb{R}^n)) \leqslant Ck^{-\beta}$$

holds for all $k \geqslant 1$.

Proof. *Step* 1: Suppose $s_1 > 0$.

Substep 1.1: The estimates from above follow from Corollary 1, Theorem 3, the multiplicativity property of the entropy numbers and some obvious monotonicity arguments.

Substep 1.2: The estimate from below will follow from Theorem 2 and the known estimates from below in case of the weighted spaces. To explain this argument observe that the following diagram

$$\begin{array}{ccc} B^{s_0}_{p_0,q_0}(\mathbb{R}_+, w_{\frac{(n-1)}{p_0}}, [1,\infty)) & \stackrel{\mathrm{ext}}{\longrightarrow} & RB^{s_0}_{p_0,q_0}(\mathbb{R}^n, [1,\infty)) \\ \mathrm{id}\downarrow & & \downarrow\mathrm{id} \\ B^{s_1}_{p_1,q_1}(\mathbb{R}_+, w_{\frac{(n-1)}{p_1}}, [1,\infty)) & \stackrel{\mathrm{tr}^*}{\longleftarrow} & RB^{s_1}_{p_1,q_1}(\mathbb{R}^n, [1,\infty)) \end{array}$$

is commutative, since the operator tr* is independent of s and p. Next we employ Franke's observation about the mapping $f \mapsto w_\alpha f$, cf. [10], [23, 5.1.3] or [7, 4.2.2]. Suppose $\alpha = (n-1)(\frac{1}{p_0} - \frac{1}{p_1})$. Then the mapping $f \mapsto w_{-\frac{(n-1)}{p_1}} f$ yields an isomorphism of $B^s_{p,q}(\mathbb{R}_+, w_\alpha, [1,\infty))$ onto $B^s_{p,q}(\mathbb{R}_+, w_{\frac{(n-1)}{p_0}}, [1,\infty))$ and of $B^s_{p,q}(\mathbb{R}_+, [1,\infty))$ onto $B^s_{p,q}(\mathbb{R}_+, w_{\frac{(n-1)}{p_1}}, [1,\infty))$ simultaneously (without restrictions on s, p and q). Hence

$$\begin{aligned} &e_k(\mathrm{id}\ : B^{s_0}_{p_0,q_0}(\mathbb{R}_+, w_\alpha, [1,\infty)) \mapsto B^{s_1}_{p_1,q_1}(\mathbb{R}_+, [1,\infty))) \\ &\quad \leqslant Ce_k(\mathrm{id}\ : B^{s_0}_{p_0,q_0}(\mathbb{R}_+, w_{\frac{(n-1)}{p_0}}, [1,\infty)) \mapsto B^{s_1}_{p_1,q_1}(\mathbb{R}_+, w_{\frac{(n-1)}{p_1}}, [1,\infty))). \end{aligned}$$

Note that

$$\delta = \left(s_0 - \frac{1}{p_0}\right) - \left(s_1 - \frac{1}{p_1}\right) > (n-1)\left(\frac{1}{p_0} - \frac{1}{p_1}\right) = \alpha \quad \text{and} \quad p_0 < p_1.$$

This means, for spaces defined on $\mathbb{R}$ we are in region III, cf. [14, Theorem 4.2] or [7, Theorem 4.3.2]. The known estimate from below is then given by

$$k^{-\beta} = k^{-\alpha + \frac{1}{p_1} - \frac{1}{p_0}} \leqslant ce_k(\mathrm{id}\ : B^{s_0}_{p_0,q_0}(\mathbb{R}, w_\alpha) \mapsto B^{s_1}_{p_1,q_1}(\mathbb{R})). \tag{19}$$

In view of this inequality it will be sufficient for us to prove that the entropy numbers of the embeddings $B^{s_0}_{p_0,q_0}(\mathbb{R}_+, w_\alpha, [1,\infty)) \hookrightarrow B^{s_1}_{p_1,q_1}(\mathbb{R}_+, [1,\infty))$ and of

$B^{s_0}_{p_0,q_0}(\mathbb{R}, w_\alpha) \hookrightarrow B^{s_1}_{p_1,q_1}(\mathbb{R})$ are asymptotically the same. But this is almost obvious. The argument looks as follows. We take $\psi_1, \psi_2 \in C^\infty(\mathbb{R})$ such that $\operatorname{supp} \psi_1 \subset (-\infty, 1/2]$, $\operatorname{supp} \psi_2 \subset [-1/2, \infty)$ and $\psi_1 + \psi_2 \equiv 1$. Then we put $\Psi_i : f \mapsto \psi_i f$, $i = 1, 2$, and $\pi : (f_1, f_2) \mapsto f_1 + f_2$. Then the following diagram

$$\begin{array}{ccc} B^{s_0}_{p_0,q_0}(\mathbb{R}, w_\alpha) & \xrightarrow{\Psi_1 \oplus \Psi_2} & B^{s_0}_{p_0,q_0}(\mathbb{R}, w_\alpha, (-\infty, 1]) \oplus B^{s_0}_{p_0,q_0}(\mathbb{R}, w_\alpha, [-1, \infty)) \\ \text{id} \downarrow & & \downarrow \text{id} \\ B^{s_1}_{p_1,q_1}(\mathbb{R}) & \xleftarrow{\pi} & B^{s_1}_{p_1,q_1}(\mathbb{R}, (-\infty, 1]) \oplus B^{s_1}_{p_1,q_1}(\mathbb{R}, [-1, \infty)) \end{array}$$

is commutative. Defining $\mathrm{id}_1 : (f_1, f_2) \mapsto (f_1, 0)$ and $\mathrm{id}_2 : (f_1, f_2) \mapsto (0, f_2)$ we have $\mathrm{id} = \mathrm{id}_1 + \mathrm{id}_2$. Thus by (19) and the commutativity of the last diagram we get

$$\begin{aligned} Ck^{-\beta} &\leqslant e_{2k}(\mathrm{id}) \leqslant e_k(\mathrm{id}_1) + e_k(\mathrm{id}_2) \\ &\leqslant 2e_k(\mathrm{id} : B^{s_0}_{p_0,q_0}(\mathbb{R}_+, w_\alpha, [1, \infty)) \mapsto B^{s_1}_{p_1,q_1}(\mathbb{R}_+, [1, \infty)). \end{aligned}$$

This finishes the proof in case s_1 positive.

Step 2: To avoid the restriction on s_1 we shall employ the properties of the operator $(\mathrm{id} - \Delta)^r$. Let $r \in \mathbb{R}$. Let $\mathscr{F}$ and $\mathscr{F}^{-1}$ denote the Fourier transform and its inverse, respectively, both defined on $\mathscr{S}'(\mathbb{R}^n)$. As it is well-known the mapping

$$I_r : f \mapsto \mathscr{F}^{-1}[(1 + |\xi|^2)^{r/2} \mathscr{F} f(\xi)]$$

yields an isomorphism of $B^s_{p,q}(\mathbb{R}^n)$ onto $B^{s-r}_{p,q}(\mathbb{R}^n)$, cf. e.g. [28, 2.3.8]. Since I_r respects radial symmetry the same happens in case of the radial subspaces. Thanks to the multiplicativity property of the entropy numbers this allows to extend our estimates obtained in Step 1 to the general situation of arbitrary s. □

There is a further scale of distribution spaces quite often investigated parallel to the Besov spaces: the Lizorkin–Triebel spaces $F^s_{p,q}$, cf. e.g. [12] or [28,29]. Again $RF^s_{p,q}(\mathbb{R}^n)$ denotes the subspace of radial distributions. These classes generalize the scale of Sobolev spaces and of Bessel potential spaces. As an immediate consequence of the preceding theorem and of the relations

$$B^s_{p,\min(p,q)}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R}^n),$$

valid without restrictions concerning the parameters s, p, q and n one obtains the following result.

Theorem 5. *Suppose* $1 \leqslant p_0 < p_1 \leqslant \infty$, $1 \leqslant q_0, q_1 \leqslant \infty$, *and* $s_0 - s_1 - n(\frac{1}{p_0} - \frac{1}{p_1}) > 0$. *Let*

$$\beta = n\left(\frac{1}{p_0} - \frac{1}{p_1}\right).$$

Then there exist constants c and C such that

$$ck^{-\beta} \leqslant e_k(\mathrm{id}\ : RF^{s_0}_{p_0,q_0}(\mathbb{R}^n) \mapsto RF^{s_1}_{p_1,q_1}(\mathbb{R}^n)) \leqslant Ck^{-\beta}$$

holds for all $k \geqslant 1$.

By using the flexibility in the third parameter q one easily derives further results.

Corollary 2. *Let* $1 \leqslant p_0 < p_1 \leqslant \infty$, $1 \leqslant q_0 \leqslant \infty$, *and* $s_0 > n(\frac{1}{p_0} - \frac{1}{p_1})$. *Let*

$$\beta = n\left(\frac{1}{p_0} - \frac{1}{p_1}\right).$$

Then there exist constants c and C such that

$$ck^{-\beta} \leqslant e_k(\mathrm{id}\ : RB^{s_0}_{p_0,q_0}(\mathbb{R}^n) \mapsto L_{p_1}(\mathbb{R}^n)) \leqslant Ck^{-\beta},$$
$$ck^{-\beta} \leqslant e_k(\mathrm{id}\ : RF^{s_0}_{p_0,q_0}(\mathbb{R}^n) \mapsto L_{p_1}(\mathbb{R}^n)) \leqslant Ck^{-\beta}$$

holds for all $k \geqslant 1$.

Remark 6. Thanks to $W^{m_0}_{p_0}(\mathbb{R}^n) = F^{s_0}_{p_0,2}(\mathbb{R}^n)$ $m_0 = s_0 \in \mathbb{N}, 1 < p_0 < \infty$ the above corollary includes the determination of the asymptotic behavior of

$$e_k(\mathrm{id}\ : RW^{m_0}_{p_0}(\mathbb{R}^n) \mapsto L_{p_1}(\mathbb{R}^n)) \sim k^{-n(\frac{1}{p_0} - \frac{1}{p_1})}$$

too.

Let $\mathrm{bmo}(\mathbb{R}^n)$ be the space of functions of local bounded mean oscillations and let $\mathrm{cmo}(\mathbb{R}^n)$ be the closure of $C_0^\infty(\mathbb{R}^n)$ in $\mathrm{bmo}(\mathbb{R}^n)$.

Corollary 3. *Let* $1 \leqslant p < \infty, 1 \leqslant q \leqslant \infty$ *and* $s > n/p$. *Let* $A^s_{p,q}(\mathbb{R}^n)$ *be either* $B^s_{p,q}(\mathbb{R}^n)$ *or* $F^s_{p,q}(\mathbb{R}^n)$. *Let X be either* $C(\mathbb{R}^n), L_\infty(\mathbb{R}^n), \mathrm{bmo}(\mathbb{R}^n)$ *or* $\mathrm{cmo}(\mathbb{R}^n)$. *Then there exist constants c and C such that*

$$ck^{-\frac{n}{p}} \leqslant e_k(\mathrm{id}\ : RA^s_{p,q}(\mathbb{R}^n) \mapsto X) \leqslant Ck^{-\frac{n}{p}}$$

holds for all $k \geqslant 1$.

Proof. The proof follows from the continuous embeddings

$$B^0_{\infty,1}(\mathbb{R}^n) \hookrightarrow \mathrm{cmo}(\mathbb{R}^n) \hookrightarrow \mathrm{bmo}(\mathbb{R}^n) \hookrightarrow B^0_{\infty,\infty}(\mathbb{R}^n),$$
$$B^0_{\infty,1}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n) \hookrightarrow B^0_{\infty,\infty}(\mathbb{R}^n)$$

and Theorems 5 and 6, respectively. □

References

[1] O.V. Besov, V.P. Ilyin, S.M. Nikol'skij, Integral Representations of Functions and Imbedding Theorems, Nauka, Moscow, 1996.
[2] G. Bourdaud, Ondelettes et espaces de Besov, Revista Mat. Iberoamericana 11 (1995) 477–512.
[3] B. Carl, Entropy numbers of diagonal operators with an application to eigenvalue problems, J. Approx. Theory 32 (1981) 135–150.
[4] B. Carl, I. Stephani, Entropy, Compactness and the Approximation of Operators, Cambridge University Press, Cambridge, 1990.
[5] F. Cobos, T. Kühn, Entropy numbers of embeddings of Besov spaces in generalized Lipschitz spaces, J. Approx. Theory 112 (2001) 73–92.
[6] S. Coleman, V. Glazer, A. Martin, Action minima among solutions to a class of Euclidean scalar field equations, Comm. Math. Phys. 58 (1978) 211–221.
[7] D.E. Edmunds, H. Triebel, Function Spaces, Entropy Numbers, Differential Operators, Cambridge University Press, Cambridge, 1996.
[8] J. Epperson, M. Frazier, An almost orthogonal radial wavelet expansion for radial distributions, J. Fourier Anal. Appl. 1 (1995) 311–353.
[9] W. Farkas, J. Johnsen, W. Sickel, Traces of anisotropic Besov–Lizorkin–Triebel spaces-a complete treatment of the borderline cases, Math. Bohemica 125 (2000) 1–37.
[10] J. Franke, Fourier-multiplikatoren, Littlewood–Paley Theoreme und Approximation durch ganze analytische Funktionen in gewichteten Funktionenräumen, Forschungsergebnisse, Univ. Jena 86/8, 1986.
[11] M. Frazier, B. Jawerth, Decomposition of Besov spaces, Indiana Univ. Math. J. 34 (1985) 777–799.
[12] M. Frazier, B. Jawerth, A discrete transform and decomposition of distribution spaces, J. Funct. Anal. 93 (1990) 34–170.
[13] D. Haroske, Embeddings of some weighted function spaces on $\mathbb{R}^n$; entropy and approximation numbers. A survey of some recent results, An. Univ. Craiova Ser. Mat. Inform. XXIV (1997) 1–44.
[14] D. Haroske, H. Triebel, Entropy numbers in weighted function spaces and eigenvalue distribution of some degenerate pseudodifferential operators. I, Math. Nachr. 167 (1994) 131–156.
[15] B. Jawerth, The trace of Sobolev and Besov spaces if $0<p<1$, Studia Math. 62 (1978) 65–71.
[16] H. König, Eigenvalue Distribution of Compact Operators, Birkhäuser, Basel, 1986.
[17] H.-G. Leopold, Embeddings and entropy numbers in Besov Spaces of generalized smoothness, in: H. Hudzik, L. Skrzypczak (Eds.), Function Spaces: The Fifth Conference, Lecture Notes in Pure and Applied Mathematics, Vol. 213, Marcel Dekker Inc., New York, 2000, pp. 323–336.
[18] P.L. Lions, Symmetrie et compacité dans les espaces de Sobolev, J. Funct. Anal. 49 (1982) 315–334.
[19] Y. Meyer, Wavelets and Operators, Cambridge University Press, Cambridge, 1992.
[20] Y. Peetre, New Thoughts on Besov Spaces, Duke University Press, Durham, 1976.
[21] A. Pietsch, Operator Ideals, North-Holland, Amsterdam, 1980.
[22] T. Runst, W. Sickel, Sobolev spaces of fractional order, Nemytskij Operators and Nonlinear Partial Differential Equations, de Gruyter, Berlin, 1996.
[23] H.-J. Schmeisser, H. Triebel, Topics in Fourier Analysis and Function Spaces, Wiley, Chichester, 1987.
[24] T. Schott, Function spaces with exponential weights. II, Math. Nachr. 196 (1998) 231–250.
[25] C. Schütt, Entropy numbers of diagonal operators between symmetric Banach spaces, J. Approx. Theory 40 (1984) 121–128.
[26] W. Sickel, L. Skrzypczak, Radial subspaces of Besov and Lizorkin-Triebel spaces: extended Strauss lemma and compactness of embeddings, J. Fourier Anal. Appl. 106 (2000) 639–662.
[27] L. Skrzypczak, Atomic decompositions on manifolds with bounded geometry, Forum Math. 10 (1998) 19–38.
[28] H. Triebel, Theory of Function Spaces, Birkhäuser, Basel, 1983.
[29] H. Triebel, Fractals and Spectra, Birkhäuser, Basel, 1997.
[30] G.N. Watson, A Treatise on the Theory of Bessel Functions, 2nd Edition, Reprinted, Cambridge University Press, Cambridge, 1995.